PSAFE: A Computationally Efficient Semi-Analytical Finite Element Scheme to Model Periodic Waveguides

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ABSTRACT

We present a Semi-Analytical Finite Element (SAFE) formulation capable of handling structural systems with spatial periodicity along the wave propagation direction. This overcomes a fundamental limitation of existing SAFE schemes, which require translational invariance along the propagation direction. The key premise of our formulation, denoted PSAFE, is the assumption of a Bloch-form solution for the unknown displacement field within the waveguide. This transforms the problem into Fourier space, replacing the spatial coordinates with their Fourier series expansion counterparts. As a result, we obtain an infinite set of coupled equations governing the waveguide's dispersion relations, each corresponding to a specific Fourier coefficient. To solve this problem numerically, we truncate the infinite series to a finite number of terms and reformulate the system as a linear eigenvalue problem. We demonstrate that accurate results can be obtained even with low truncation orders and that increasing the system's dimensionality improves accuracy, albeit at the cost of higher computational expense.

INTRODUCTION

Wave propagation in thin structural elements such as plates plays a critical role in applications ranging from non-destructive evaluation (NDE) to structural health monitoring (SHM). In recent years, this area has also attracted interest from the metamaterials community, where periodic arrays of resonators coupled to waveguides enable wave manipulation phenomena like bandgaps and mode conversion [1]. Capturing these effects requires numerical methods that can resolve both the structural periodicity and the wave–resonator interactions. Common strategies include Bloch Operator FEM (BOFEM) [2] and Wave FEM (WFEM) [3], which discretize the unit cell and apply Bloch conditions [4]. While effective, these approaches can be computationally intensive, hence reduced order models have also been developed [5, 6].

A major improvement in computational efficiency could be achieved by discretizing only the cross-sectional domain and assuming an analytical solution along the wave

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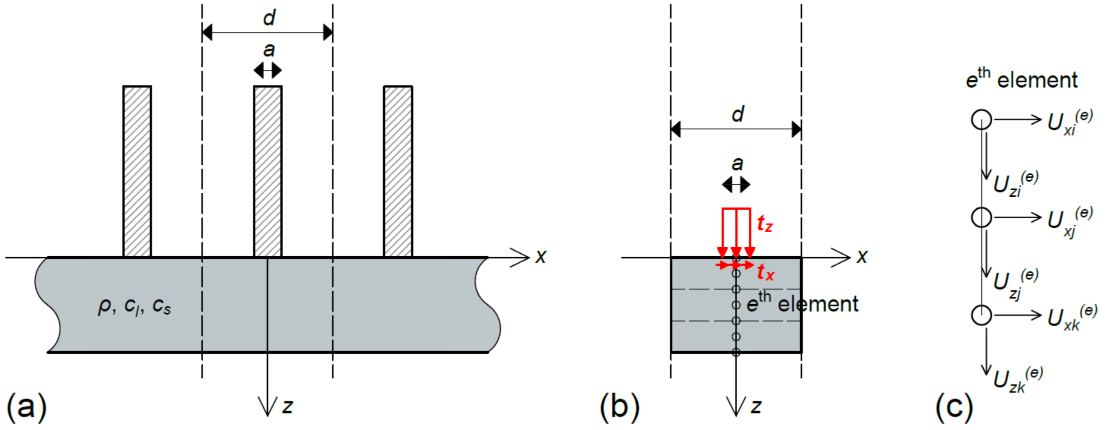

Figure 1. Schematic of a plate coupled to an infinite array of equally spaced resonant structures at its upper surface. (a) General physical scenario. (b) Unit cell of the problem, with the resonant structure represented by the tractions it exerts on the plate. (c) Degrees of freedom for a one-dimensional three-node element.

propagation direction, as is typically done in Semi-Analytical Finite Element (SAFE) formulations [7–10]. However, the existing SAFE formulations require translational invariance along the wave propagation direction, making them unsuitable for spatially periodic structures.

The full description of the method and results presented here can be found in Ref [11], where the SAFE approach is extended to efficiently model waveguides coupled to periodic resonant structures. The proposed formulation (i) applies Bloch conditions to the displacement field, (ii) discretizes only the waveguide's cross-section, and (iii) represents resonators through periodic mechanical impedances. This approach retains the periodic nature of the system while reducing computational costs. The numerical example shown here demonstrates the method's accuracy in capturing dispersion phenomena in plates coupled to mass-spring-dashpot resonators.

PSAFE FORMULATION

Let us consider the wave propagation in a plate periodically coupled to resonant structures attached along its top surface, as shown in Fig. 1(a). The plate is modeled as an isotropic, linear viscoelastic material with density ρ and wave speeds c_l and c_s. The resonators are identical, span the out-of-plane direction, and are spaced by a constant pitch d along the wave propagation direction x. Under these conditions, the system is analyzed in the x-z plane assuming plane strain. A single unit cell, consisting of a plate segment of length d with one resonator, defines the periodic geometry.

The proposed formulation models the wave propagation solely through the dynamic equilibrium of the plate, using a tailored Semi-Analytical Finite Element method denoted PSAFE ("Periodic" SAFE). The resonators are modeled as impedance-type loads, applying displacement-dependent tractions on the plate's upper surface, as illustrated in Fig. 1(b). While these tractions are idealized as uniform in the figure, their actual spatial form varies with the resonator model. Assuming time-harmonic motion at frequency ω, the displacement field satisfies Bloch conditions and is expressed as a spatial Fourier expansion:

$$\mathbf{u}(x,z)=\sum_{n=-\infty}^{+\infty}\hat{\mathbf{u}}_n(z)\mathrm{e}^{\mathrm{i}n\bar{\xi}x}\mathrm{e}^{\mathrm{i}\xi x},\quad \hat{\mathbf{u}}_n(z)=\frac{1}{d}\int_{-d/2}^{d/2}\mathbf{u}(x,z)\mathrm{e}^{-\mathrm{i}n\bar{\xi}x}\,\mathrm{d}x \tag{1}$$

where $\bar{\xi}=2\pi/d$ is the modulation wavenumber, ξ is the unknown wavenumber(s) in the first Brillouin zone [12] at the specific frequency ω and $\hat{\mathbf{u}}_n(z)$ is the n-th Fourier coefficient of $\mathbf{u}(x,z)$.

The cross-sectional domain of the plate lying on the y-z plane, Ω, is discretized through n_{el} three-node one-dimensional finite elements (as shown in Fig. 1(c)), each having domain Ω_e. This results in a total of $n_{\text{nodes}}=2\times n_{\text{el}}+1$ nodes and $n_{\text{dof}}=2\times n_{\text{nodes}}$ degrees-of-freedom. The variation of the generic n-th coefficient $\hat{\mathbf{u}}_n(z)$ of the displacement field $\mathbf{u}^{(e)}(x,z)$ along the z-direction over the generic Ω_e is expressed as:

$$\hat{\mathbf{u}}_n^{(e)}(z)=\mathbf{N}(z)\hat{\mathbf{U}}_n^{(e)} \tag{2}$$

where the vector $\hat{\mathbf{U}}_n^{(e)}$ collects the component-wise n-th Fourier coefficients of the nodal displacement vector $\mathbf{U}^{(e)}(x)=\begin{bmatrix}U_{xi} & U_{zi} & U_{xj} & U_{zj} & U_{xk} & U_{zk}\end{bmatrix}^T$ (T being the transpose operator) and $\mathbf{N}(z)$ is the shape function matrix.

After various algebraic manipulations, which are detailed in [11], the following system of equations, which governs the plate's dynamics, is finally obtained:

$$\left[\mathbf{K}_1+\mathrm{i}\xi_h\mathbf{K}_2+\xi_h^2\mathbf{K}_3-\omega^2\mathbf{M}\right]\hat{\mathbf{U}}_h-\sum_{p=-\infty}^{+\infty}\hat{\mathbf{Z}}_p(\omega)\hat{\mathbf{U}}_{h-p}=\mathbf{0},\quad \text{with } h\in\mathbb{Z} \tag{3}$$

where $\xi_h=\xi+h\bar{\xi}$ is the total wavenumber expressed as the contribution of the wavenumber in the first Brillouin zone ξ and that of the modulation wavenumber $\bar{\xi}$; $\mathbf{K}_1$, $\mathbf{K}_2$, $\mathbf{K}_3$ and $\mathbf{M}$ are $n_{dof}\times n_{dof}$ stiffness and mass matrices; $\hat{\mathbf{U}}_n$ is the n-th Fourier coefficients of the n_{dof} global nodal displacement vector $\mathbf{U}(x)$, and $\hat{\mathbf{Z}}_p(\omega)$ is the p-th Fourier coefficient of the $n_{dof}\times n_{dof}$ global impedance matrix $\mathbf{Z}(x,\omega)$, which is introduced to express the tractions exerted by the resonant structures. As a matter of example, for an array of mass-spring-dashpot oscillators with mass m_o, stiffness k_o and damping coefficient c_o, $\hat{\mathbf{Z}}_p(\omega)$ can be written as [11]:

$$\hat{\mathbf{Z}}_p(\omega)=\begin{bmatrix}0 & 0 & \cdots & 0\\ 0 & \frac{m_o\omega^2(\mathrm{i}\omega c_o+k_o)}{-\omega^2 m_o+\mathrm{i}\omega c_o+k_o} & \cdots & 0\\ \vdots & \vdots & \ddots & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\frac{1}{d} \tag{4}$$

An approximate expression for Eq. (3) can be obtained by setting $|n|\leq P$ and $|p|\leq 2P$, where P is an integer number, hence obtaining:

$$\left[\mathbf{K}_1+\mathrm{i}\xi_h\mathbf{K}_2+\xi_h^2\mathbf{K}_3-\omega^2\mathbf{M}\right]\hat{\mathbf{U}}_h-\sum_{p=-2P}^{+2P}\hat{\mathbf{Z}}_p(\omega)\hat{\mathbf{U}}_{h-p}=\mathbf{0},\quad \text{with } |h|\leq P \tag{5}$$

For a given real and positive angular frequency ω, Eq. (5) can be seen as a quadratic generalized eigenvalue problem of size $(2P+1) \times n_{\text{dof}}$ with respect to the complex wavenumber $\xi_h = \xi + h\bar{\xi}$. Eq. (5) can be linearized by reformulating it into a first-order eigensystem, thereby doubling its algebraic size, as:

$$\left(\begin{bmatrix} \mathbf{0} & \mathbf{A}_0 \\ \mathbf{A}_0 & \mathbf{A}_1 \end{bmatrix} - \xi \begin{bmatrix} \mathbf{A}_0 & \mathbf{0} \\ \mathbf{0} & -\mathbf{A}_2 \end{bmatrix} \right) \begin{bmatrix} \hat{\mathbf{\Phi}} \\ \xi \hat{\mathbf{\Phi}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{6}$$

where $\hat{\mathbf{\Phi}} = \begin{bmatrix} \hat{\mathbf{U}}_{-P} & \ldots & \hat{\mathbf{U}}_0 & \ldots & \hat{\mathbf{U}}_{+P} \end{bmatrix}^T$, while the quantities $\mathbf{A}_0$, $\mathbf{A}_1$, and $\mathbf{A}_2$ are given as:

$$\begin{aligned} \mathbf{A}_0 &= \mathbf{I}_{2P+1} \otimes \mathbf{K}_1 + \mathrm{i}\bar{\xi}\mathbf{D} \otimes \mathbf{K}_2 + \bar{\xi}^2 \mathbf{D}^{\circ 2} \otimes \mathbf{K}_3 - \omega^2 \mathbf{I}_{2P+1} \otimes \mathbf{M} - \Xi(\omega), \\ \mathbf{A}_1 &= \mathrm{i}\mathbf{I}_{2P+1} \otimes \mathbf{K}_2 + 2\bar{\xi}\mathbf{D} \otimes \mathbf{K}_3, \\ \mathbf{A}_2 &= \mathbf{I}_{2P+1} \otimes \mathbf{K}_3 \end{aligned} \tag{7}$$

where $\otimes$ indicates Kronecker product; $\mathbf{I}_k$ is the identity matrix of size k; $\mathbf{D} = \mathrm{diag}(-P, -P+1, \ldots, 0, \ldots, P-1, P)$; $\circ$ indicates Hadamard (element-wise) product; and Ξ is a $(2P+1) \times (2P+1)$ matrix arranged as:

$$\Xi(\omega) = \begin{bmatrix} \hat{\mathbf{Z}}_0(\omega) & \hat{\mathbf{Z}}_{-1}(\omega) & \cdots & \hat{\mathbf{Z}}_{-2P}(\omega) \\ \hat{\mathbf{Z}}_{+1}(\omega) & \hat{\mathbf{Z}}_0(\omega) & \cdots & \hat{\mathbf{Z}}_{-2P+1}(\omega) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\mathbf{Z}}_{+2P}(\omega) & \hat{\mathbf{Z}}_{+2P-1}(\omega) & \cdots & \hat{\mathbf{Z}}_0(\omega) \end{bmatrix} \tag{8}$$

EXAMPLE: PLATE COUPLED TO AN ARRAY OF MASS-SPRING-DASHPOT OSCILLATORS

Let us consider a 1 mm-thick steel plate modeled as a linear elastic material with density $\rho = 7700$ kg/m^3 and wave speeds $c_l = 5960$ and $c_s = 3260$ m/s. The plate cross-section is discretized using five three-node elements (11 nodes, 22 DOFs). The plate is coupled with an array of discrete mass-spring-dashpot oscillators with mass $m_o = 1.54$ g, damping coefficient $c_o = 0$, and resonant frequency $f_o = 1$ MHz, which are regularly spaced along the x-direction with an inter-distance $d = 2$ mm.

The dispersion relations for the oscillators-coupled plate are described by Eq. (5) where the Fourier coefficients of the global impedance matrix are those given by Eq. (4). Solutions are obtained by solving the corresponding eigenvalue problem expressed in Eq. (6) for a chosen truncation parameter $P = 20$, i.e., by prescribing an angular frequency ω and solving for the corresponding complex wavenumbers ξ. The real and imaginary parts of the corresponding wavenumbers are plotted with black dots in Fig. 2(a) and Fig. 2(b), respectively. For validation purposes, Fig. 2(a) also shows with orange circles the results offered by the WFEM method applied on an analogous 2D finite element plane-strain model of the coupled oscillators-plate system, whose implementation details can be found in [11]. The two sets of results are essentially superposed, thus confirming the accuracy of the PSAFE's solution.

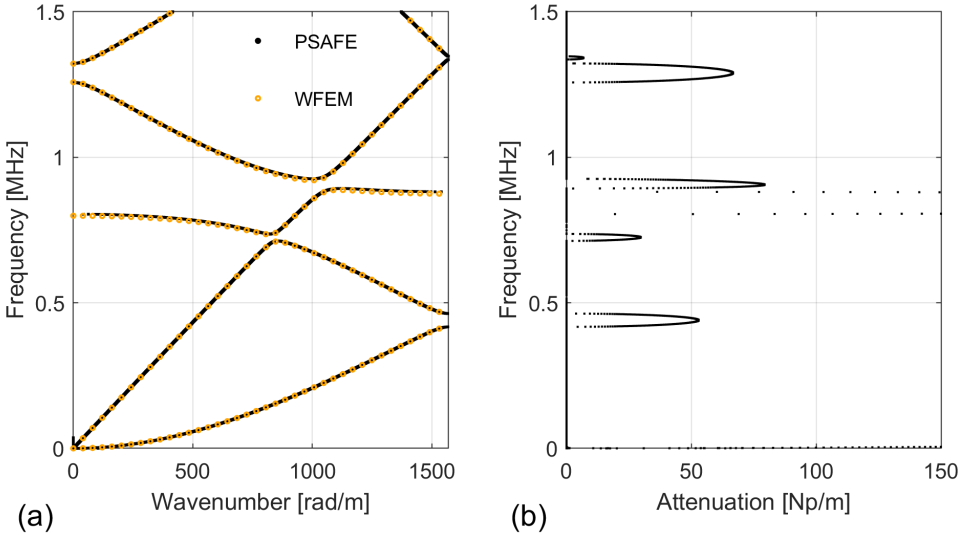

Figure 2. Dispersion relations of a 1 mm-thick steel plate coupled to an array of mass-spring-dashpot oscillators, spaced 2 mm apart, on its top surface. (a) Real and (b) imaginary parts of the complex wavenumbers corresponding to plane waves in the system.

CONCLUSIONS

This work presented a Semi-Analytical Finite Element formulation (PSAFE) for modeling waveguides coupled to periodic resonant structures. By enforcing Bloch periodicity on the displacement field and modeling the resonators as periodic mechanical impedances, the authors derived an infinite set of coupled equations in Fourier space governing wave dispersion in the system. A numerical solution was obtained by truncating these equations and reformulating the problem as a linear eigenvalue system, which only requires discretizing the cross-section of the waveguide, thereby reducing computational cost.

The formulation was demonstrated through a case study involving a steel plate coupled to discrete mass-spring-dashpot resonators. Dispersion curves computed using PSAFE were validated against results from the Wave Finite Element Method, showing excellent agreement.

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