A Study on Guided Wave Propagation in Quasicrystal Plates

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ABSTRACT

Guided wave-based Structural Health Monitoring (SHM) is a highly promising approach for assessing the structural integrity of large-scale engineering systems. This method offers significant advantages, including rapid and extensive area monitoring, long-distance wave propagation, the capability to detect internal damage, and low energy consumption. Quasicrystals are recent advancements in materials characterized by low frictional and adhesion coefficients, along with high abrasion and thermal resistance. These properties make them highly suitable for coating surfaces in various engineering applications. In quasicrystals, the quasi-periodic atomic arrangement introduces additional phason displacement modes alongside phonon modes, presenting a unique challenge in guided wave analysis. In this work, the semi-analytical finite element (SAFE) method has been implemented for computing dispersion curves, phase and group velocities for quasicrystals. These methods have advantage in wave propagation analysis for complex cross section where analytical solution is difficult to obtain. Using the SAFE, a novel eigenvalue formulation has been developed for established quasicrystal models in literature: Bak's model and Lubensky's model. This study focuses on understanding how phason modes behave in Lamb wave analysis and also understanding how phonon-phason coupling constant has effect on Lamb wave analysis. Results have been obtained for various phason elastic constants reported in literature.

INTRODUCTION

The presence of damage or defects in the engineered structure is critical to safety, as even minor cracks can lead to catastrophic failure, risking lives and property. This concern has led to extensive research on non-destructive evaluation (NDE) and its evolution into Structural Health Monitoring (SHM), which focuses on real-time damage detection and life prediction with minimal human involvement. SHM methods are generally classified as vibration-based or guided wave-based, with guided ultrasonic waves proving especially effective due to their ability to inspect large areas quickly, detect internal damage, and operate efficiently and economically [1]. However, they pose analytical

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challenges due to their multimodal, dispersive nature, and frequency-dependent spatial attenuation, which is crucial in selecting modes with minimal loss for large-area inspections.

This work explores the use of Quasicrystals in Lamb wave analysis. Quasicrystals represent a unique class of solids distinct from crystals and amorphous materials. The first quasicrystal, discovered by Shechtman et al. [2] in an Al-Mn alloy, exhibited five-fold rotational symmetry, previously considered forbidden in crystals due to the loss of translational symmetry and long-range order. Subsequent studies revealed quasicrystals with other symmetries, including eight-fold, ten-fold, and twelve-fold. Their distinctive atomic arrangement imparts unique physical properties such as low friction, high thermal resistance, and significant abrasion resistance, making them promising for engineering applications.

Quasicrystals exhibit quasi-periodic atomic arrangements with long-range order [3]. This introduces a new degree of freedom: the phason displacement mode, in addition to the conventional phonon mode. The dimensionality of quasi-periodicity defines whether the structure is 1D, 2D, or 3D. Theories like phason flip and diffusion models were given in literature to describe phason displacement mode in quasicrystal. Chellepan [4] demonstrated that the phason mode could be used for control of wave propagation. In literature, guided wave studies involving phason modes are limited. Zhang [5] used Legendre polynomials for wave analysis in 1D graded quasicrystal plates, though the method is not suitable for general cross section.

This work focuses on Lamb wave propagation analysis of quasicrystal plate using Semi Analytical Finite Element Method (SAFE). SAFE has gained significant popularity in recent years for addressing wave propagation challenges in waveguides. Developed as an alternative to conventional techniques like the global matrix method, SAFEM is particularly advantageous for solving waveguide problems with arbitrary cross sections, as highlighted by Hayashi, Song, and Rose [6]. This method involves discretizing the wave guide cross section while employing an analytical solution in the direction of wave propagation. Using a variational scheme, a system of linear equations is formulated with frequency and wave number as unknowns. These unknowns are then solved using standard eigenvalue solving techniques.

PHYSICAL BASIS OF ELASTICITY OF QUASICRYSTAL:

In quasicrystal due to its quasi-periodic atomic arrangement, an additional displacement mode exists known as phason mode. So, total two kind of displacement mode exists, U and W, the former being phonon displacement mode and latter being phason mode. Phason mode exists because, to completely specify diffraction experiment results of quasicrystal, number of basis vectors required exceeds the number of physical basis, that is three [7].

This point suggests that quasi-periodic structure can be visualized as periodic in higher dimensional space. This description is known as super-space description [3]. So the total displacement field for quasicrystal can be expressed as :

$$\bar{\mathbf{u}} = \mathbf{u}^{\parallel} \oplus \mathbf{u}^{\perp} = \mathbf{u} \oplus \mathbf{w}$$

Here, $\oplus$ represents direct sum. For quasicrystal, u which is phonon displacement field, lies in physical space or parallel space $E^3_{\parallel}$, whereas w which is phason displacement,lies in perpendicular space $E^3_{\perp}$ which is an internal space. Furthermore, both phonon and phason modes depends only on parallel space $\mathbf{r}^{\parallel}$ that is:

$$\mathbf{u} = \mathbf{u}(\mathbf{r}^{\parallel}), \quad \mathbf{w} = \mathbf{w}(\mathbf{r}^{\parallel}).$$

Quasicrystals are classified based on dimensionality as number of dimensions in which quasi-periodicty exists. 1D, 2D, and 3D quasicrystals have quasi-periodicity in one, two, and three directions, respectively.

DEFORMATION TENSORS:

Because of having two displacement modes namely phonon and phason, we have two different strains. Gradient of phonon vector u is decomposed into two parts, ε_{ij} and ω_{ij}. ε_{ij} is due to deformation and ω_{ij} is due to rigid rotation. whereas,phason strain can also be decomposed into symmetric and anti-symmetric component, but since phason physically signifies local atomic rearrangement contributing to only deformation and not rigid rotation. Hence, phason strain, w_{ij} is asymmetric tensor which can be expressed as :

$$\nabla \mathbf{w} = w_{ij} = \frac{\partial w_i}{\partial x_j} \tag{1}$$

ELASTO-HYDRODYMANIC EQUATION:

In contrast to general elasto-dynamics, for quasicrystal the governing equation is called elasto-hydrodynamics because the equation of motion of phonon mode are elasto dynamic equation while the motion of phason are diffusion equation originated from hydrodynamics. From free energy density equation, we have :

$$\sigma_{ij} = \frac{\partial F}{\partial \varepsilon_{ij}} = C_{ijkl}\varepsilon_{kl} + R_{ijkl}w_{kl} \text{ , H}_{ij} \quad = \frac{\partial F}{\partial w_{ij}} = K_{ijkl}w_{kl} + R_{klij}\varepsilon_{kl} \tag{2}$$

We can represent this equation in matrix form as :

$$\begin{bmatrix} \sigma_{ij} \\ H_{ij} \end{bmatrix} = \begin{bmatrix} [\mathbf{C}] & [\mathbf{R}] \\ [\mathbf{R}]^T & [\mathbf{K}] \end{bmatrix} \begin{bmatrix} \varepsilon_{ij} \\ w_{ij} \end{bmatrix} \tag{3}$$

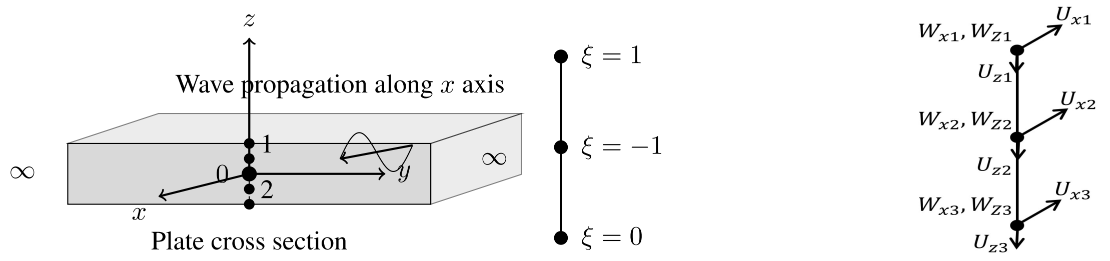
Here, C_{ijkl} denotes the phonon elastic constant, K_{ijkl} the phason elastic constant, and R_{ijkl} the phonon-phason coupling constant.

SAFE FORMULATION OF QUASICRYSTAL:

The Strain-Displacement relationship for both phonon and phason fields can be represented as :

$$\boldsymbol{\varepsilon_e} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial z} & 0 & 0 \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial x} & 0 & 0 \end{bmatrix} \begin{bmatrix} u_x \\ u_z \\ w_x \\ w_z \end{bmatrix} \quad \text{and} \quad \boldsymbol{\omega}_e = \begin{bmatrix} 0 & 0 & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial z} \\ 0 & 0 & \frac{\partial}{\partial z} & 0 \\ 0 & 0 & 0 & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} u_x \\ u_z \\ w_x \\ w_z \end{bmatrix} \tag{4}$$

Here $\boldsymbol{\varepsilon_e} = \begin{bmatrix}\varepsilon_{xx} & \varepsilon_{zz} & \varepsilon_{xz}\end{bmatrix}^T$, $\boldsymbol{\omega_e} = \begin{bmatrix}\omega_{xx} & \omega_{zz} & \omega_{xz} & \omega_{zx}\end{bmatrix}^T$



(a) Infinite plate with 1D element along thickness

(b) Nodes along ξ direction.

(c) DOF for 1D 3 noded element

Figure 1. Illustration of plate wave propagation and node points along ξ-axis.

Equation (4), containing phonon strain $\boldsymbol{\varepsilon_e}$ can be expressed as :

$$\boldsymbol{\varepsilon_e} = \begin{bmatrix}\varepsilon_{xx} & \varepsilon_{zz} & \varepsilon_{xz}\end{bmatrix}^{\mathrm{T}} = \begin{bmatrix}\boldsymbol{L_x}\frac{\partial}{\partial x} + \boldsymbol{L_z}\frac{\partial}{\partial y}\end{bmatrix}[u] \tag{5}$$

and phason strain can be expressed as :

$$\boldsymbol{\omega_e} = \begin{bmatrix}\omega_{xx} & \omega_{zz} & \omega_{xz} & \omega_{zx}\end{bmatrix}^T = \left[\boldsymbol{L'_x}\frac{\partial}{\partial x} + \boldsymbol{L'_z}\frac{\partial}{\partial y}\right][u]. \tag{6}$$

Here $\boldsymbol{L_x}$, $\boldsymbol{L_z}$, $\boldsymbol{L_x}'$ and $\boldsymbol{L'_z}$ can be expressed as matrix. In Semi-Analytical Finite Element Method(SAFE), analytical solution is assumed in wave propagating direction while discretisation is done in the direction perpendicular to the wave propagating direction.For this set of SAFE formulation for quasicrystal, we assume propagating wave in X direction for both phonon and phason so we have:

$$\tilde{U} = \boldsymbol{N(z)} q_e e^{-\mathrm{i}(Kx-\omega t)} \tag{7}$$

Here $\boldsymbol{N(z)}$ is shape function for the 3 noded element where each node contains 4 degree of freedom, 2 each in phonon X and Z direction and phason X and Z direction. So, on substituting displacement field, we obtain expression for phonon strain and phason strain as :

$$\begin{aligned} \boldsymbol{\epsilon_e} &= \left[\boldsymbol{L_x}\frac{\partial}{\partial x} + \boldsymbol{L_z}\frac{\partial}{\partial z}\right] \boldsymbol{N(z)} \cdot q_e \cdot e^{-\mathrm{i}(Kx-\omega t)}, \\ \boldsymbol{\omega_e} &= \left[\boldsymbol{L'_x}\frac{\partial}{\partial x} + \boldsymbol{L'_z}\frac{\partial}{\partial z}\right] \boldsymbol{N(z)} \cdot q_e \cdot e^{-\mathrm{i}(Kx-\omega t)}. \end{aligned} \tag{8}$$

phonon and phason strain in Equation (8) can also be expressed as :

$$\begin{aligned}\boldsymbol{\epsilon_e} &= \left[\boldsymbol{L_z N_{,z}} - iK\boldsymbol{L_x N}\right] q_e \cdot e^{-\mathrm{i}(Kx-\omega t)} = \left[\boldsymbol{B_1} - iK\boldsymbol{B_2}\right] q_e \cdot e^{-\mathrm{i}(Kx-\omega t)} \\ \boldsymbol{\omega_e} &= \left[\boldsymbol{L_z}'\boldsymbol{N_{,z}} - iK\boldsymbol{L_x}'\boldsymbol{N}\right] q_e \cdot e^{-\mathrm{i}(Kx-\omega t)} = \left[\boldsymbol{B_1}' - iK\boldsymbol{B_2}'\right] q_e \cdot e^{-\mathrm{i}(Kx-\omega t)}\end{aligned} \tag{9}$$

where $\boldsymbol{B_1} = \boldsymbol{L_z N_{,z}}$, $\boldsymbol{B_2} = \boldsymbol{L_x N}$, $\boldsymbol{B_1}' = \boldsymbol{L_z}'\boldsymbol{N_{,z}}$ and $\boldsymbol{B_2}' = \boldsymbol{L_x}'\boldsymbol{N}$

EQUATION OF MOTION OF QUASICRYSTAL USING SAFE

The energy functional (π) will be:

$$\pi = \int_\Omega F\,\mathrm{d}\Omega - \int_\Omega \left[(f_i - \rho\ddot{u}_i)\,u_i + (g_i - \kappa\dot{w}_i)\,w_i\right]\mathrm{d}\Omega - \int_{S_l} (T_i u_i + h_i w_i)\,\mathrm{d}S \tag{10}$$

Here, κ is the kinetic coefficient of the phason field. In absence of external traction and internal force we have :

$$\delta\pi = \int_{t_1}^{t_2}\int_\Omega \left[\underset{\text{(I)}}{\delta\varepsilon^{\mathsf{T}}\boldsymbol{C}\varepsilon} + \underset{\text{(II)}}{\delta\omega^{\mathsf{T}}\boldsymbol{K}\omega} + \underset{\text{(III)}}{\delta\varepsilon^{\mathsf{T}}\boldsymbol{R}\omega} + \underset{\text{(IV)}}{\delta\omega^{\mathsf{T}}\boldsymbol{R'}\varepsilon} + \underset{\text{(V)}}{\delta u^{\mathsf{T}}\rho\ddot{u}} + \underset{\text{(VI)}}{\delta w^{\mathsf{T}}\kappa\dot{w}}\right]\mathrm{d}\Omega\,\mathrm{d}t \tag{11}$$

This equation (11) has got six parts and the difference in Bak's and Lubensky's model will come only in part VI of this equation. If we expand each part by substituting the value of U, W, $\boldsymbol{\varepsilon_e}$ and $\boldsymbol{\omega_e}$ we will get elemental level equation, and if we combine we will construct elemental eigen value problem as:

$$\int_{t_1}^{t_2}\left\{\bigcup_{e=1}^{n_{el}} \delta\mathbf{q}_e^T\left[\boldsymbol{k_1^{(e)}} + ik\mathbf{k}_2^{(e)} + k^2\mathbf{k}_3^{(e)} - \omega^2\mathbf{m}^{(e)} + i\omega\mathbf{m}_1^{(e)}\right]\mathbf{q}_e\right\}dt = 0 \tag{12}$$

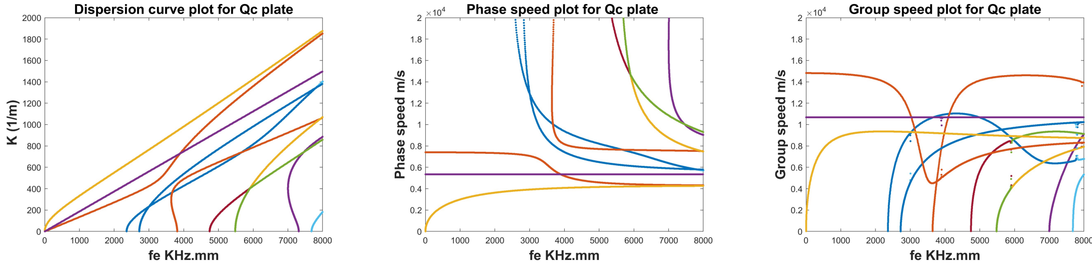
For arbitrary $\delta\mathbf{q}_e^T$ the final symmetric global SAFE eigenvalue equation is obtained, from which dispersion relation, phase and group speed is computed and, is given as:.

$$\left[\mathbf{K_1} + \mathbf{k}\hat{\mathbf{K}}_\mathbf{2} + \mathbf{k^2K_3} - \omega^\mathbf{2}\mathbf{M} + \omega\mathbf{M_1}\right]\hat{\mathbf{U}} = \mathbf{0} \tag{13}$$

SAFE RESULTS FOR QUASICRYSTAL PLATE:

In this work phason elasticity constants k_1 and k_2 are varied for Al-Ni-Co and Al-Pd-Mn quasicrystal as reported in the literature [8, 9], obtained by various measuring techniques to see how phason modes are affected by varying k_1 and k_2 for the analysis of Lamb waves of a quasicrystal plate. There has been wide variation in the reported value of k_1 and k_2 obtained by various measuring techniques. Plate thickness is taken as 1 mm and quasicrystal properties are taken from [9] as reported for various decagonal and icosahedral quasicrystal. In all the cases, the coupling constant R is varied as a function of phonon elastic constant M.

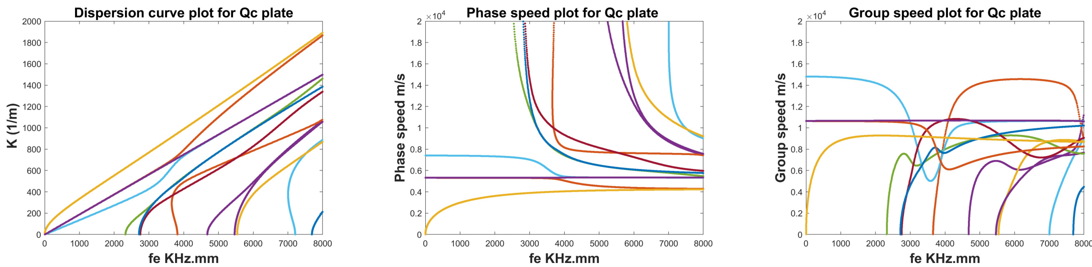
CASE I : COUPLED QUASICRYSTAL (R/M =0.001), k_1=124GPa, k_2=24GPa



(a) Dispersion curve (b) Phase speed (c) Group speed

Figure 2. Dispersion, phase speed, and group speed plots for $N = 4$.

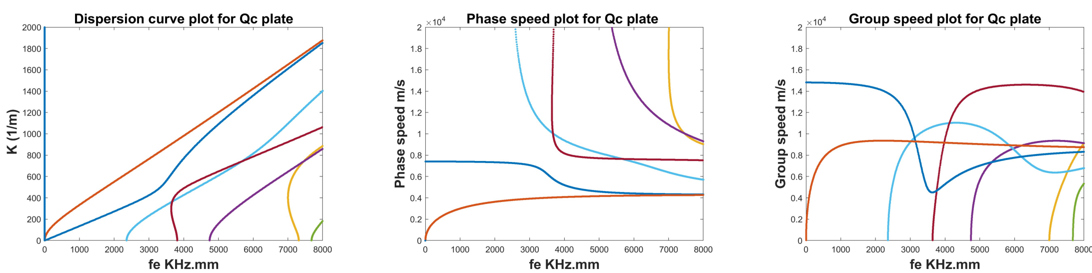
CASE II : COUPLED QUASICRYSTAL (R/M =0.1), k_1=124GPa, k_2=24GPa



(a) Dispersion curve (b) Phase speed (c) Group speed

Figure 3. Dispersion, phase speed, and group speed plots for $N = 4$.

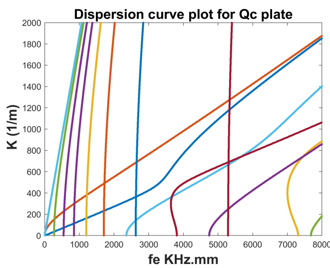
CASE III : UNCOUPLED QUASICRYSTAL (R/M =0), k_1= k_2=0



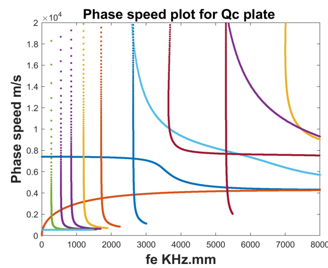
(a) Dispersion curve (b) Phase speed (c) Group speed

Figure 4. Dispersion, phase speed, and group speed plots for $N = 4$.

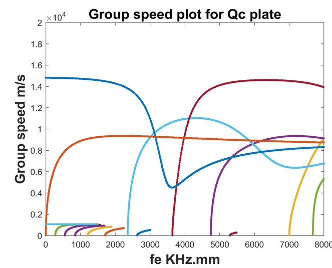
CASE IV : COUPLED QUASICRYSTAL (R/M =0), k_1=1.24GPa, k_2=.24GPa



(a) Dispersion curve



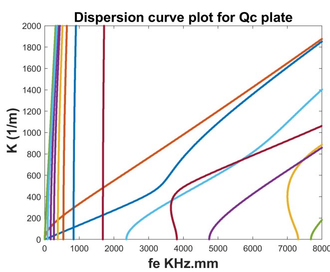
(b) Phase speed



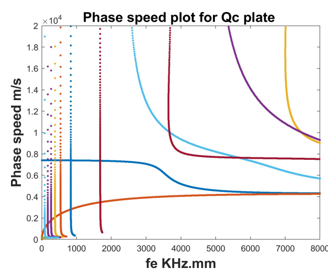
(c) Group speed

Figure 5. Dispersion, phase speed, and group speed plots for $N = 4$.

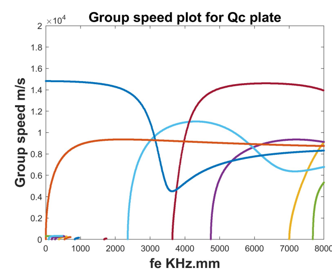
CASE V : COUPLED QUASICRYSTAL (R/M =0), k_1=124MPa, k_2=-50MPa



(a) Dispersion curve



(b) Phase speed



(c) Group speed

Figure 6. Dispersion, phase speed, and group speed plots for $N = 4$.

DISCUSSION

In cases I and II, both phonon and phason modes appear in the dispersion relation, as well as in the group speed and phase speed plots. However, when the coupling constant and phason elastic constant are set to zero, the phason mode disappears, leaving only the phonon mode. Furthermore, when the phonon elastic constant is set to the value corresponding to aluminium, the SAFE results precisely match those of aluminium in the literature [10]. Upon varying the values of k1 and k2, from higher GPa to lower MPa orders, it is observed that only the phason modes shift, while the phonon mode remains unchanged. Additionally, coupling between the phonon and phason modes is observed at higher coupling constants, leading to failures in the mode sorting algorithm due to mode crossings in cases I and II.

CONCLUDING REMARKS

In this study, we observe that variations in the coupling constant and phason elastic constant as documented in the literature, lead to notable changes in phase and group velocities, particularly at higher coupling constant values. Given the diffusive nature of phason modes, these findings suggest promising applications in layered structures for aerospace engineering. Such configurations will lead to energy leakage, further reducing phase and group velocities while introducing attenuation effects. Consequently, future investigations would focus on the potential of layered quasicrystal materials and conduct detailed leaky Lamb wave analyses to harness these phenomena effectively.

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