Uncertainty Laws of Free-Vibration Modal Identification

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ABSTRACT

Free vibration test is a commonly used method in structural condition assessment to acquire vibration data from which modal parameters (e.g., natural frequencies, damping ratios, and mode shapes) can be extracted. This naturally leads to a fundamental question: how a field test can be planned optimally? Focusing on modal identification under free vibration, the paper addresses this question by proposing uncertainty laws, which give closed-form asymptotic formulas for coefficients of variation of modal parameters. These laws characterize the relationship between the identification uncertainties of modal parameters and the test configuration (e.g., location, number and noise level of sensors). With the assumptions of long data and small damping, the derivation involves the Fisher information matrix evaluation. Theoretical results and assumptions are validated using synthetic data. According to the derived uncertainty laws, the key factors governing identification uncertainty include the damping ratio, the bandwidth factor, the modal signal-to-noise ratio, and the effective data length. The developed uncertainty laws provide a scientific basis for managing identification uncertainties in free vibration tests.

INTRODUCTION

Structural modal parameters (e.g., natural frequencies, damping ratios and mode shapes) are crucial in the field of structural health monitoring, as they determine the structural response to dynamic loads of wind, earthquake and human activity. These parameters can be identified from vibration data collected through ambient, forced and free vibration tests [1]. Among these methods, ambient vibration test is popular due to

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its cost-effectiveness and ease of implementation. However, it often yields identification results with high uncertainty due to the unmeasured and moderate nature of ambient excitations. Forced vibration test can significantly reduce the identification uncertainty by increasing and measuring the structural input. Nevertheless, this approach is often impractical because of the high cost of the required excitation equipment. Free vibration test offers a feasible alternative, as it can produce high signal-to-noise ratio (s/n) data without the need for measuring the input force.

In a free vibration test, a structure is artificially excited and then allowed to vibrate without a continued external force. While the artificial excitation is required, its exact measurement is unnecessary. Various modal identification methods have been developed based on free vibration data [2–4]. Notably, Bayesian approaches can quantify the uncertainty associated with the identified modal parameters. However, they typically provide only numerical indicators (e.g., variance) based on the specific dataset and do not offer insights into how test configurations influence the uncertainty, nor how to reduce it.

To address this issue, uncertainty laws have been developed for ambient [5,6] and shaker-based forced vibration tests [7,8]. These laws provide closed-form asymptotic expressions for the posterior coefficients of variation (c.o.v.s) of modal parameters, thereby characterizing the key factors governing identification uncertainty. They not only provide insights into the achievable precision for a given test configuration but also serve as a valuable tool for test configuration optimization [9].

This paper develops uncertainty laws for modal identification under free vibration, with both known and unknown initial conditions (ICs). Section 2 introduces a Bayesian modal identification algorithm based on the finite-time Fourier transform (FTFT). Section 3 presents an overview of the derived uncertainty laws. In Section 4, a synthetic example is used to validate the proposed laws and to explore the impact of key factors. Finally, conclusions are summarized in Section 5.

BAYESIAN MODAL IDENTIFICATION UNDER FREE VIBRATION

Let $\hat{\ddot{\mathbf{y}}}(t) \in \mathbb{R}^n$ be the structural acceleration time history at n instrumented degrees of freedom (DoFs). Its scaled Fourier transformation over a finite time window is defined as

$$\hat{\ddot{\mathbf{Y}}}(\mathrm{f}) = \frac{1}{\sqrt{T}}\int_0^T \hat{\ddot{\mathbf{y}}}(t)\, e^{-\mathbf{i}2\pi \mathrm{f}t} dt \qquad (1)$$

where $\hat{\ddot{\mathbf{Y}}}(\mathrm{f}) \in \mathbb{C}^n$, $\mathbf{i}$ is the imaginary unit and T is the sampling duration. By discretizing Eqn. (1) with a sampling interval Δt (sec), one can obtain the discretized FTFT (abbreviated as DFTFT):

$$\hat{\ddot{\mathbf{Y}}}_k = \sqrt{\frac{\Delta t}{N}}\sum_{j=0}^{N-1} \hat{\ddot{\mathbf{y}}}_j \exp\left[-\frac{2\pi \mathbf{i} kj}{N}\right] \quad (k = 0, \dots, N-1) \qquad (2)$$

Here, $\{\hat{\ddot{\mathbf{y}}}_j \in \mathbb{R}^n : j = 0,1,\dots,N-1\}$ denotes the sampled acceleration time-domain data with N samples. For $k \le N_q$, where N_q is the index corresponding to the Nyquist frequency, $\hat{\ddot{\mathbf{Y}}}_k$ corresponds to the frequency $\mathrm{f}_k = k/(N\Delta t)$ (Hz).

Consider identifying a classically damped and well-separated mode on a narrow frequency band, the measured DFTFT within the band is modeled as

$$\hat{\ddot{\boldsymbol{Y}}}_k = \boldsymbol{\varphi}\mathcal{F}_{\ddot{\eta}}(\mathrm{f_k}) + \boldsymbol{\varepsilon}_k \tag{3}$$

where $\boldsymbol{\varphi} \in \mathbb{R}^n$ denotes the partial mode shape confined to the instrumented DoFs; $\boldsymbol{\varepsilon}_k \in \mathbb{C}^n$ accounts for measurement noise and modeling error at frequency f_k; $\mathcal{F}_{\ddot{\eta}}(\mathrm{f_k})$ is the DFTFT of modal acceleration response given by

$$\mathcal{F}_{\ddot{\eta}}(\mathrm{f_k}) = h_k^u u + h_k^v v \tag{4}$$

where

$$h_k^u = \frac{\mathbf{i}2\pi f\beta_k}{1-\beta_k^2-\mathbf{i}2\zeta\beta_k}, h_k^v = \frac{\beta_k^2+\mathbf{i}2\zeta\beta_k}{1-\beta_k^2-\mathbf{i}2\zeta\beta_k} \tag{5}$$

represent the frequency response functions (FRFs) with respect to scaled initial modal displacement u and velocity v, respectively. Substituting Eqns. (4)~(5) into Eqn. (3) gives

$$\hat{\ddot{\boldsymbol{Y}}}_k = \boldsymbol{\varphi}\boldsymbol{H}_k\boldsymbol{\psi} + \boldsymbol{\varepsilon}_k \tag{6}$$

where $\boldsymbol{H}_k = [h_k^u, h_k^v\,] \in \mathbb{C}^{1\times 2}$ is the FRF vector, and $\boldsymbol{\psi} = [u, v\,]^{\mathrm{T}} \in \mathbb{R}^{2\times 1}$ contains the scaled ICs.

Given the structural acceleration data $\{\hat{\ddot{\boldsymbol{y}}}_j\}$, the goal of modal identification is to estimate the unknown parameter set $\boldsymbol{\theta} = \{f, \zeta, \boldsymbol{\varphi}, \boldsymbol{\psi}, S_e\}$. Note that this reduces to $\boldsymbol{\theta} = \{f, \zeta, \boldsymbol{\varphi}, S_e\}$ if the ICs are known. A probabilistic model is constructed below to address this problem. First, the error $\boldsymbol{\varepsilon}_k$ is assumed to follow a complex normal distribution with zero mean and covariance matrix of $S_e\boldsymbol{I}_n$, i.e., $\boldsymbol{\varepsilon}_k \sim \mathcal{CN}(0, S_e\boldsymbol{I}_n)$. Consequently, $\hat{\ddot{\boldsymbol{Y}}}_k$ follows a complex normal distribution with the mean $\boldsymbol{\varphi}\boldsymbol{H}_k\boldsymbol{\psi}$ and the covariance matrix $S_e\boldsymbol{I}_n$. Assuming statistical independence of $\boldsymbol{\varepsilon}_k$ across different frequencies leads to $p\left(\left\{\hat{\ddot{\boldsymbol{Y}}}_k\right\}\middle|\boldsymbol{\theta}\right) = \prod_{k=1}^{N_f} p\left(\hat{\ddot{\boldsymbol{Y}}}_k\middle|\boldsymbol{\theta}\right)$. Adopting a uniform prior distribution for unknown parameters $\boldsymbol{\theta}$, the posterior probability density function (PDF) is given by Bayes' theorem as $p\left(\boldsymbol{\theta}\middle|\left\{\hat{\ddot{\boldsymbol{Y}}}_k\right\}\right) \propto p\left(\left\{\hat{\ddot{\boldsymbol{Y}}}_k\right\}\middle|\boldsymbol{\theta}\right) = \exp[-\mathcal{L}(\boldsymbol{\theta})]$, where the negative log-likelihood function (NLLF) $\mathcal{L}(\boldsymbol{\theta})$ is expressed as

$$\mathcal{L}(\boldsymbol{\theta}) = nN_f\ln\pi + nN_f\ln S_e + S_e^{-1}\sum_{k=1}^{N_f}\left[\hat{\ddot{\boldsymbol{Y}}}_k - \boldsymbol{\varphi}\boldsymbol{H}_k\boldsymbol{\psi}\right]^*\left[\hat{\ddot{\boldsymbol{Y}}}_k - \boldsymbol{\varphi}\boldsymbol{H}_k\boldsymbol{\psi}\right] \tag{7}$$

At this stage, the parameter set $\boldsymbol{\theta}$ is non-identifiable since one can obtain identical NLLFs when multiplying $\boldsymbol{\varphi}$ by a constant c and then dividing it by $\boldsymbol{\psi}$. To make the problem physically identifiable, a norm constraint of $\boldsymbol{\varphi}^{\mathrm{T}}\boldsymbol{\varphi} = 1$ is further introduced here. An analytical expression for the posterior PDF $p\left(\boldsymbol{\theta}\middle|\left\{\hat{\ddot{\boldsymbol{Y}}}_k\right\}\right)$ does not exist due to the complicated nature of NLLF. With sufficient data, the Laplace method in Bayesian inference can be applied to approximate the posterior using a multivariate normal distribution, whose mean vector and covariance matrix are given by

$$\hat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}} \mathcal{L}(\boldsymbol{\theta}); \quad \hat{\boldsymbol{\Sigma}} = [\nabla^2\mathcal{L}(\boldsymbol{\theta})_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}}]^{-1} \tag{8}$$

Here, the operator '$\arg\min_{\boldsymbol{\theta}}$ [·]' denotes the value of $\boldsymbol{\theta}$ that minimizes NLLF. The mean vector $\hat{\boldsymbol{\theta}}$ corresponds to the maximum a posteriori estimate and the covariance matrix $\hat{\boldsymbol{\Sigma}}$, quantifying the remaining uncertainties of modal parameters, is given by the inverse of the Hessian of the NLLF at $\hat{\boldsymbol{\theta}}$. For simplicity, the above Bayesian modal identification method will be referred to as 'BAYEMA-Free' hereafter.

TABLE I. KEY RESULTS OF UNCERTAINTY LAWS

Parameter $\boldsymbol{x}$	zeroth order law δ_{x0}^2	Data length factor $B_x(\kappa)$	Unknown-IC factor $c_x(\kappa)$	Known-IC factor $c'_x(\kappa)$
Frequency f	$\frac{\zeta}{2\pi N_c B_f(\kappa)}$	$\frac{2}{\pi}k_1$	$\frac{4k_0}{k_2}$	$\frac{2k_1}{k_2}$
Damping ζ	$\frac{1}{2\pi\zeta N_c B_\zeta(\kappa)}$	$\frac{2}{\pi}\left(k_2-\frac{2k_0^2}{\kappa}\right)$	$\frac{4k_0\left(k_2-\frac{2k_0^2}{\kappa}\right)}{k_1k_2}$	$\frac{2\left(k_2-\frac{2k_0^2}{\kappa}\right)}{k_2}$
Mode shape $\overline{\boldsymbol{\varphi}}$	$\frac{n-1}{2\pi\zeta N_c B_{\overline{\varphi}}(\kappa)}$	$\frac{2}{\pi}k_0$	1	1
Channel noise PSD S_e	$\frac{1}{(n-1)N_f B_{S_e}(\kappa)}$	1	$\frac{n-1}{n}\gamma_{free}$	$\frac{n-1}{n}\gamma_{free}$
Initial displacement u	$\frac{1}{2\pi\zeta N_c B_u(\kappa)}$	$\frac{2}{\pi}k_1$	2	
Initial velocity v	$\frac{1}{2\pi\zeta N_c B_v(\kappa)}$	$\frac{2}{\pi}k_1$	2	

Note:
1. Symbols f, ζ, $\overline{\boldsymbol{\varphi}}$, u, v, and S_e denote the 'true' value of modal properties rather than the variable in the likelihood function;
2. Mode shape c.o.v. is defined as the square root sum of eigenvalues of the covariance matrix of $\overline{\boldsymbol{\varphi}}$;
3. Define $k_0=\tan^{-1}\kappa$, $k_1=\left(\tan^{-1}\kappa-\frac{\kappa}{\kappa^2+1}\right)$, $k_2=\left(\tan^{-1}\kappa+\frac{\kappa}{\kappa^2+1}\right)$

OVERVIEW OF UNCERTAINTY LAWS

Within the context of BAYEMA-Free method, the mode of interest is assumed to be well-separated within a narrow frequency band and is characterized by its natural frequency f (in Hz), damping ratio ζ, unit-norm mode shape $\overline{\boldsymbol{\varphi}}$ $(=\boldsymbol{\varphi}/\sqrt{\boldsymbol{\varphi}^{\mathrm{T}}\boldsymbol{\varphi}})$ as well as initial modal displacement u and velocity v. The frequency band for modal identification is selected with a range of $f(1\pm\kappa\zeta)$, where κ is a dimensionless bandwidth factor. Assuming the dynamic responses are measured over a duration T_d with the sampling frequency f_s, the selected band contains $N_f = 2\kappa\zeta f T_d$ frequency points and the "effective data length" is defined here as $N_c = T_d f$.

Under the above definitions, it can be shown that for long data $N_f \gg 1$ and small damping $\zeta \ll 1$, the (squared) posterior c.o.v. δ_x^2 of the modal parameter x is asymptotically given by

$$\delta_x^2 \sim \delta_{x,free}^2 = \delta_{x0}^2\frac{c_x(\kappa)}{\gamma_{free}} \quad \text{for } N_f \gg 1, \zeta \ll 1 \tag{9}$$

where $\delta_{x,free}^2$ defines the uncertainty laws for modal identification under free vibration; the symbol "$\sim$" indicates that the ratio of both sides tends to 1 under asymptotic conditions; The term δ_{x0}^2 (except for mode shape and ICs) corresponds to the "zeroth-order law" in the ambient case [1]; the modal s/n ratio γ_{free} is defined as

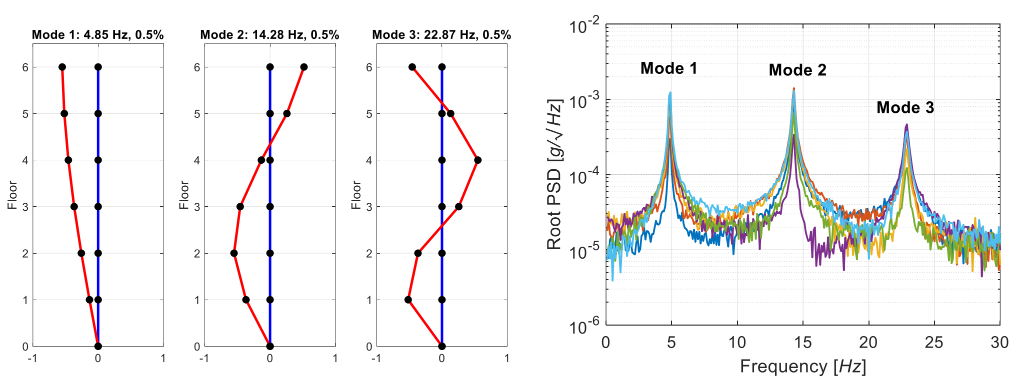
$$\gamma_{free} = \frac{4\pi^2 f^2 u^2 + v^2}{4S_e\zeta^2} \tag{10}$$

As indicated by Eqn.(10), the uncertainty law $\delta_{x,free}^2$ decreases as the modal s/n ratio γ_{free} increases and can be made arbitrarily small. This can be achieved by using higher-quality sensors (i.e., smaller noise PSD S_e) or applying stronger excitation to generate larger values of ICs. Note that γ_{free} is evaluated to set $u = 0$ for $x = v$ and $v = 0$ for $x = u$, reflecting their decoupled nature in identification uncertainty. The quantity $c_x(\kappa)$ is defined as the "unknown-IC factor", specific for the context of free vibration test with unknown ICs. For the case of known ICs, the uncertainty laws retain the same form as in Eqn. (10), except that $c_x(\kappa)$ is replaced by the "known-IC factor" $c_x'(\kappa)$.

The above expressions for different parameters are summarized in Table I. The derivation takes advantage of the assumptions of long data asymptotics, small damping asymptotics and asymptotic decoupling [1]. Due to the space limitations, the detailed derivation is omitted, and an empirical verification is provided in the next section.

EMPIRICAL VERIFICATION

To validate the proposed uncertainty laws, this section presents an illustrative example based on synthetic data. The considered structure is a six-story shear-type building. A simplified finite element model is constructed using Euler beam elements, with a lumped mass of 1×10^3 kg and a flexural stiffness of 1.6×10^4 kN/m. Only the first three modes are considered in the analysis. Their corresponding natural frequencies and mode shapes are shown in Figure 1 (a). The damping ratio is uniformly set to 0.5% for all modes. To comprehensively validate the proposed uncertainty laws, three cases are considered, as summarized in Table II. Each case involves six independent tests, resulting in a total of 18 tests for analysis and discussion. In all cases, the structure is subjected to an ideal impact force at the initial time, and then vibrates freely. The structural responses recorded after one second following the application of the impact force are used for analysis, ensuring the presence of non-zero initial conditions and the elimination of external force influence. A typical root power spectral density (PSD) of structural responses is shown in Figure 1(b) for visualization purposes.



(a) Modal model (b) Typical Root PSD of output

Figure 1 Six-story shear-type building, synthetic data

TABLE II. THREE CONSIDERED TEST CONFIGURATIONS

Case ID	PSD of error term $\left(\times 10^{-5}\ \mathrm{g}/\sqrt{\mathrm{Hz}}\right)$	Excitation location	Duration (sec)
1	$2^{0:5}$	6	30
2	1	1:6	30
3	1	6	20:5:45

Direct verification

The derived uncertainty laws are first verified by comparing the identification uncertainties calculated by the BAYEMA-Free method and predicted by the corresponding uncertainty laws. For modal identification, the selected bands used for the modal identification are [4.56, 5.14] Hz, [13.42, 15.13] Hz, and [21.50, 24.24] Hz with $f_0(1 + \kappa\zeta_0)$, where κ equals to 6, ζ_0 equals to 1% and f_0 equals to 4.85 Hz, 14.28, and 22.87 Hz respectively.

The scatterplots of posterior c.o.v.s of modal parameters identified by BAYEMA-Free method versus those predicted by corresponding uncertainty laws are shown in Figure 3. It is observed that all markers are distributed around the diagonal lines, indicating a strong agreement between the BAYEMA-Free method and corresponding uncertainty laws. This demonstrates that the proposed laws can effectively capture the identification uncertainty of modal parameters. In addition, it has been confirmed that knowing the ICs significantly reduces the identification uncertainties of natural frequency and damping ratio, while having negligible effect on mode shapes.

Effect of modal s/n ratio

Case 1 investigates the effect of modal s/n ratio on identification uncertainty. In this case, we simply assume that the structural responses are measured by accelerometers with varying noise levels under the same external forces applied at floor 6. The resulting posterior c.o.v.s of the identified modal parameters are shown in Figure 3. Consistent with the theoretical results from Eqn. (10), the identification uncertainty vanishes when increasing the modal s/n ratio γ. This confirms that employing sensors with lower noise levels can effectively reduce the uncertainty in modal identification.

Effect of data length

Case 2 examines the effect of test duration on identification uncertainty. According to the derived uncertainty laws, the posterior c.o.v.s are expected to decrease with an increase N_c. As shown in Figure 4, the c.o.v.s exhibit a diminishing trend as N_c increases. However, in practical applications, N_c cannot grow indefinitely due to rapid energy dissipation, which is a distinguishing feature of free vibration tests compared to forced or ambient vibration tests. Obviously, the uncertainty laws link identification uncertainty to the effective data length, thereby providing a practical means to determine the required time duration to achieve a desired level of uncertainty.

Effect of excitation location

Case 3 explores the impact of excitation location on the identification uncertainty. Although excitation location is not explicitly parameterized in the derived uncertainty laws, it exerts a notable influence by affecting the modal s/n. In this case, excitation is sequentially applied at locations from floor 1 to floor 6. As shown in Figure 5, the identification uncertainty is highly sensitive to the excitation location. A key observation is that the uncertainty is inversely proportional to the values of the mode shapes at the excitation point.

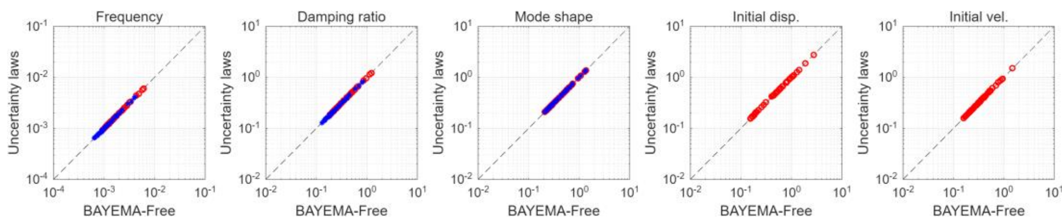


Figure 2 Posterior c.o.v. calculated by BAYEMA-Free method and predicted by uncertainty laws

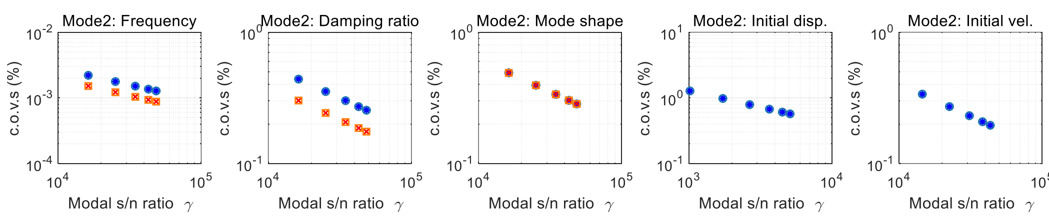


Figure 3 Effect of modal s/n ratio on c.o.v. of modal parameters
(Blue star: BAYEMA-Free, unknown ICs; Blue circle: uncertainty laws, unknown ICs Red cross: BAYEMA-Free, known ICs; Red square: uncertainty laws, known ICs)

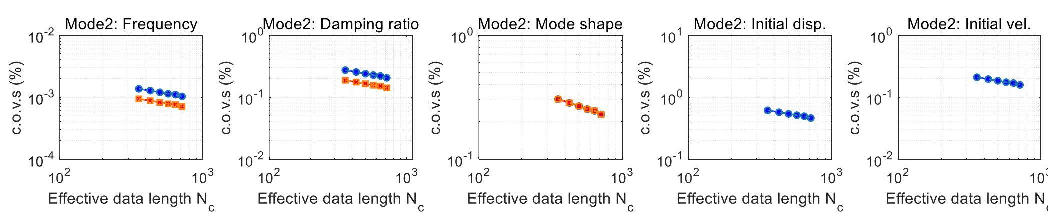


Figure 4 Effect of effective data length on c.o.v.s of modal parameters
(Labels have the same meaning as in Figure 3)

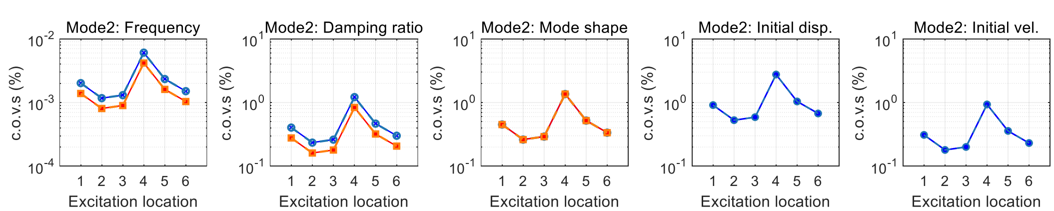


Figure 5 Effect of excitation location on c.o.v.s of modal parameters.
(Labels have the same meaning as in Figure 3)

CONCLUSION

This paper presents the uncertainty laws under free vibration, i.e., the closed-form asymptotic expressions for the c.o.v. of modal parameters. The key results have been summarized in Table I, with both cases of known and unknown ICs. The uncertainty laws have been verified by synthetic data. In the context of free vibration test, the identification uncertainties for frequencies, damping ratios, mode shapes as well as initial modal displacement and velocity are governed by the following dimensionless parameters: damping ratio, bandwidth factor, modal s/n ratio and effective data length. Additionally, knowing the ICs can effectively reduce the identification uncertainty of natural frequency and damping ratio, but has no influence on that of mode shapes.

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